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Interaction of vortices and waves in stratified turbulence

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joint work with
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Similarity between MHD and stratified turbulence

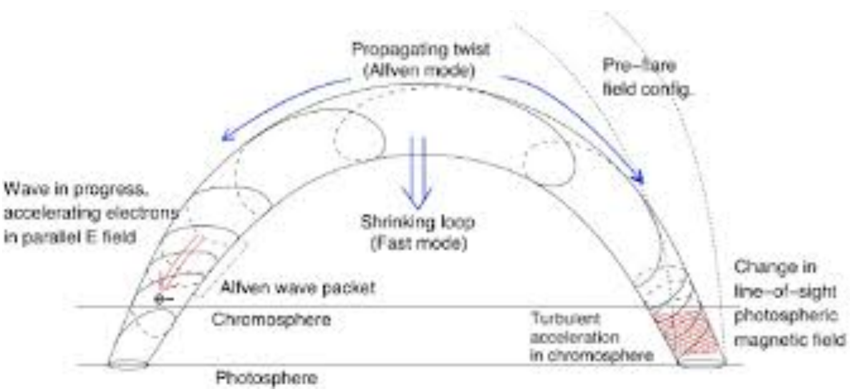
MHD : velocity field is coupled with ***magnetic field*** by the Lorentz force

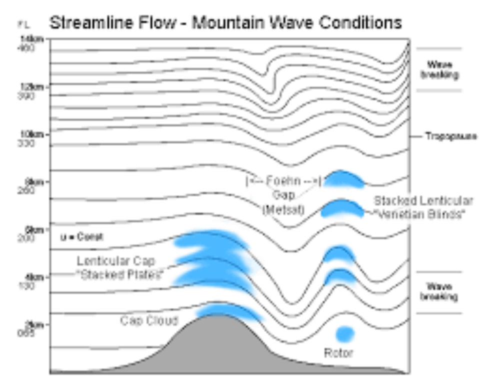
stratified : velocity field is coupled with ***density*** by buoyancy

characteristic waves exist in turbulence

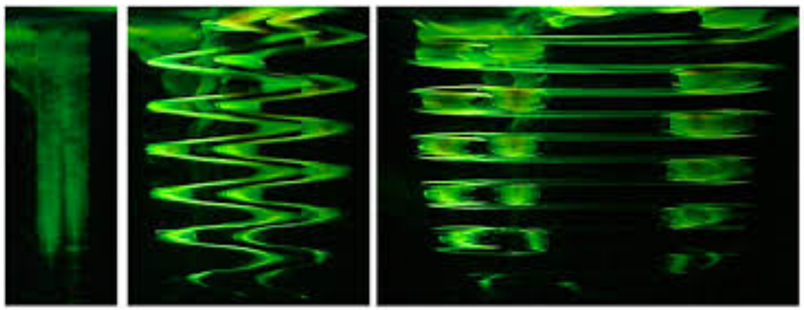
MHD
Alfvén waves

stratified turbulence
gravity waves



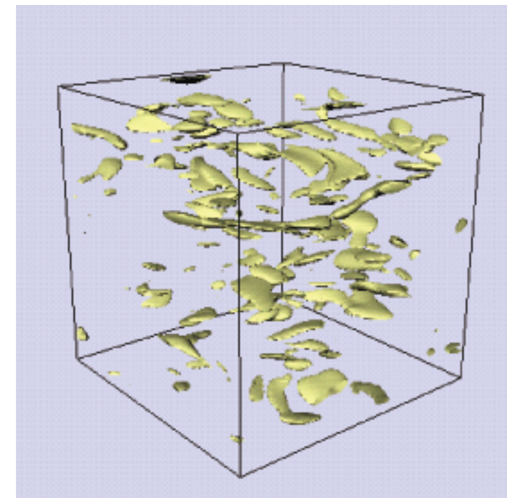


Vortices in stably stratified turbulence



Zig-Zag instability:
Three-dimensional stability of a vertical columnar vortex pair in a stratified fluid.
Billant, P. & Chomaz, J.-M.
J. Fluid Mech. **419**, 65–91(2000).

Produced from "large" scales,
(starting from vertical columnar vortices)



Scattered pancakes:
Diffusion in stably stratified turbulence.
Kimura, Y. & Herring, J.R.
J. Fluid Mech. **328**, 253–269(1996).

Produced from "small" scales
(starting from random isotropic vortices)

Q: Are they really different things?

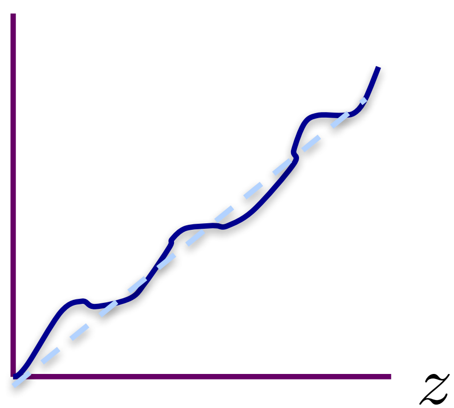
Navier-Stokes equation with the Boussinesq approximation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \theta\hat{\mathbf{z}} + \mathbf{f}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = \kappa\nabla^2\theta - N^2 w$$

$$\nabla\cdot\mathbf{u} = 0$$

$\Theta(z) = N^2 z + \theta$



where

$\mathbf{u} = (u,\ v,\ w)$: velocity

θ : temperature fluctuations

$N^2 = \frac{g\alpha}{T_0}\frac{\partial \bar{T}}{\partial z}$: Brunt - Väisälä frequency

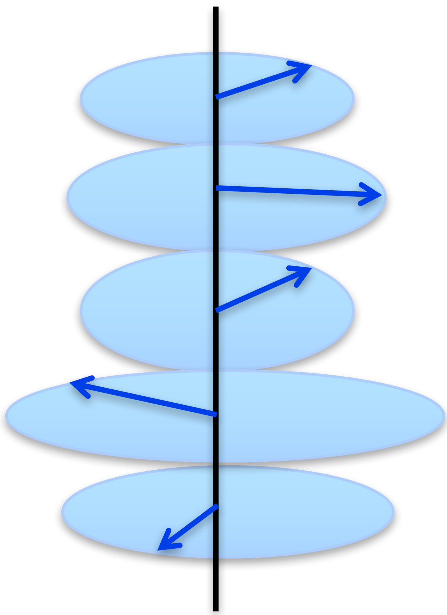
$\mathbf{f}$: External forcing (in fourier space)

Numerical Methods

- forced simulations
- 2π-periodic box with $512^3 \sim 2048^3$ grid points ($R_\lambda \sim 400$)
- 3^{rd} order time-marching scheme
- Initial energy spectrum : $E(k) = 0$
- Force horizontal velocity components
- Add red noise to modes within a wave number band ($k_f \sim 4, 10$)
- Two types of 2D forcing (quasi 2D, pure 2D)

Two types of 2D forcing

quasi 2D

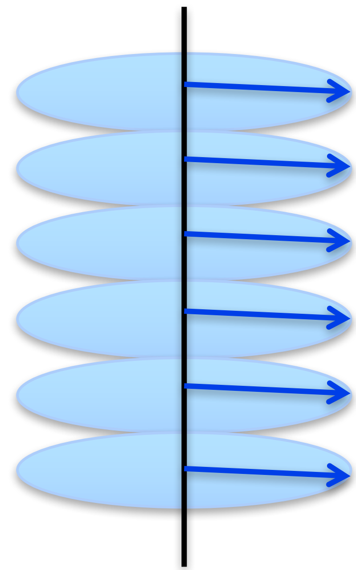


$$\mathbf{f} = (f(k_x, k_y, k_z), g(k_x, k_y, k_z), 0)$$

Forcing component may differ vertically

Inputting seeds of gravity waves

pure 2D



$$\mathbf{f} = (f(k_x, k_y, 0), g(k_x, k_y, 0), 0)$$

Purely 2-dimensional

Gravity waves are generated by the coupling between velocity and temperature fluctuations

Craya-Herring decomposition

$$\nabla \cdot \mathbf{u} = 0 \quad \Longleftrightarrow \quad \mathbf{k} \cdot \tilde{\mathbf{u}} = 0$$

incompressibility

$\tilde{\mathbf{u}}$ is spanned by two independent vectors perpendicular to $\mathbf{k}$

$$\mathbf{e}_1(\mathbf{k}) = \frac{\mathbf{k} \times \hat{\mathbf{z}}}{\|\mathbf{k} \times \hat{\mathbf{z}}\|} = \frac{1}{\sqrt{k_x^2 + k_y^2}} \begin{pmatrix} k_y \\ -k_x \\ 0 \end{pmatrix}$$

$$\mathbf{e}_2(\mathbf{k}) = \frac{\mathbf{k} \times \mathbf{k} \times \hat{\mathbf{z}}}{\|\mathbf{k} \times \mathbf{k} \times \hat{\mathbf{z}}\|} = \frac{1}{\sqrt{k_x^2 + k_y^2 + k_z^2}\sqrt{k_x^2 + k_y^2}} \begin{pmatrix} k_z k_x \\ k_z k_y \\ -(k_x^2 + k_y^2) \end{pmatrix}$$

$$\mathbf{e}_3(\mathbf{k}) = \frac{\mathbf{k}}{\|\mathbf{k}\|} = \frac{1}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$$

orthnormal coordinates

$$\tilde{\mathbf{u}}(\mathbf{k}) = \phi_1 \mathbf{e}_1(\mathbf{k}) + \phi_2 \mathbf{e}_2(\mathbf{k})$$

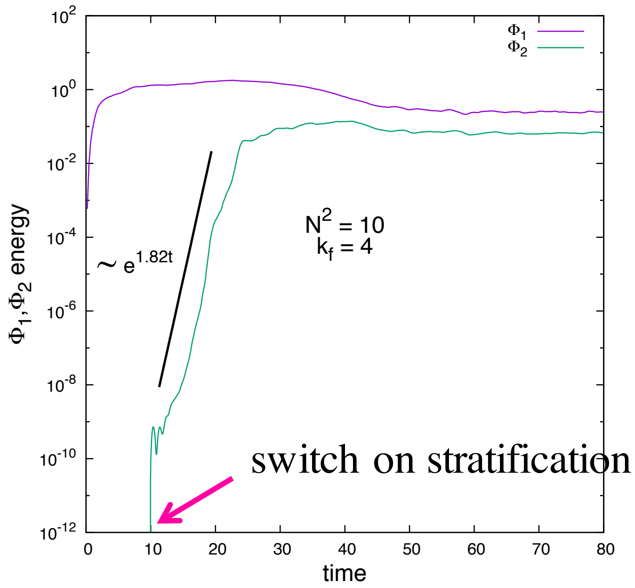
$$\phi_1 = \tilde{\mathbf{u}}(\mathbf{k}) \cdot \mathbf{e}_1(\mathbf{k}) = \frac{1}{\sqrt{k_x^2 + k_y^2}}(k_y \tilde{u} - k_x \tilde{v}) = \frac{i}{\sqrt{k_x^2 + k_y^2}}\tilde{\omega}$$

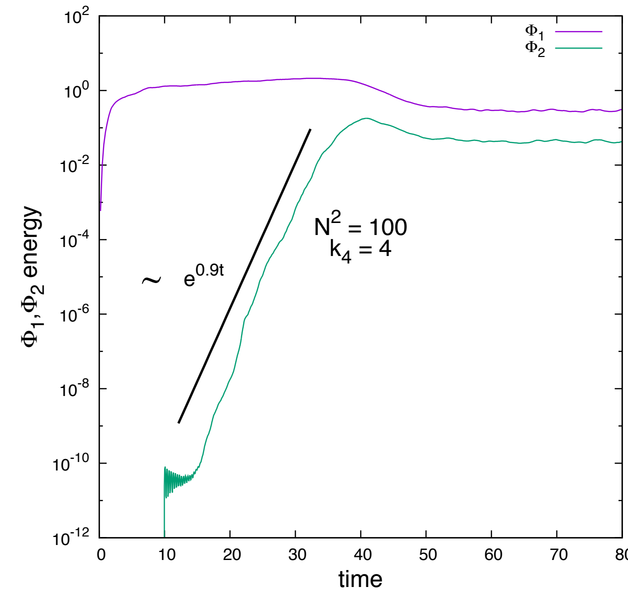
(vortical)

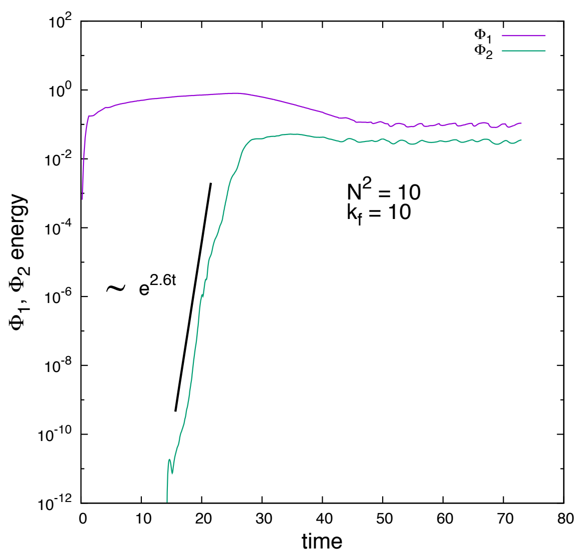
$$\phi_2 = \tilde{\mathbf{u}}(\mathbf{k}) \cdot \mathbf{e}_2(\mathbf{k}) = \frac{\sqrt{k_x^2 + k_y^2 + k_z^2}}{\sqrt{k_x^2 + k_y^2}} \tilde{w}$$

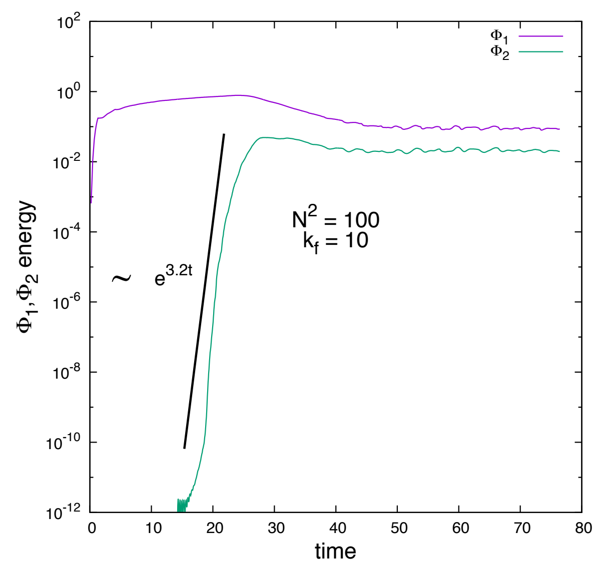
(wavy)

Growth of Φ_1 and Φ_2 energy

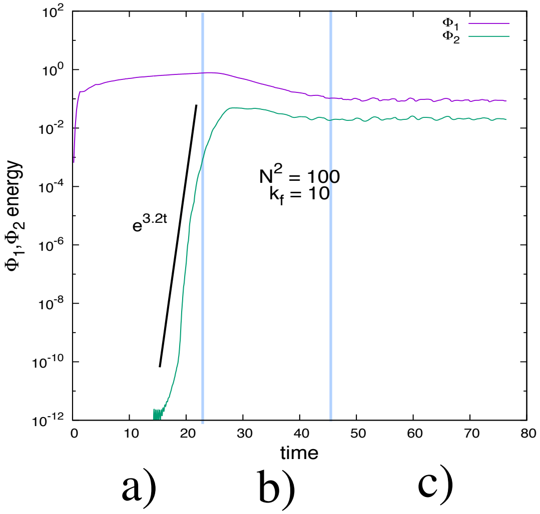




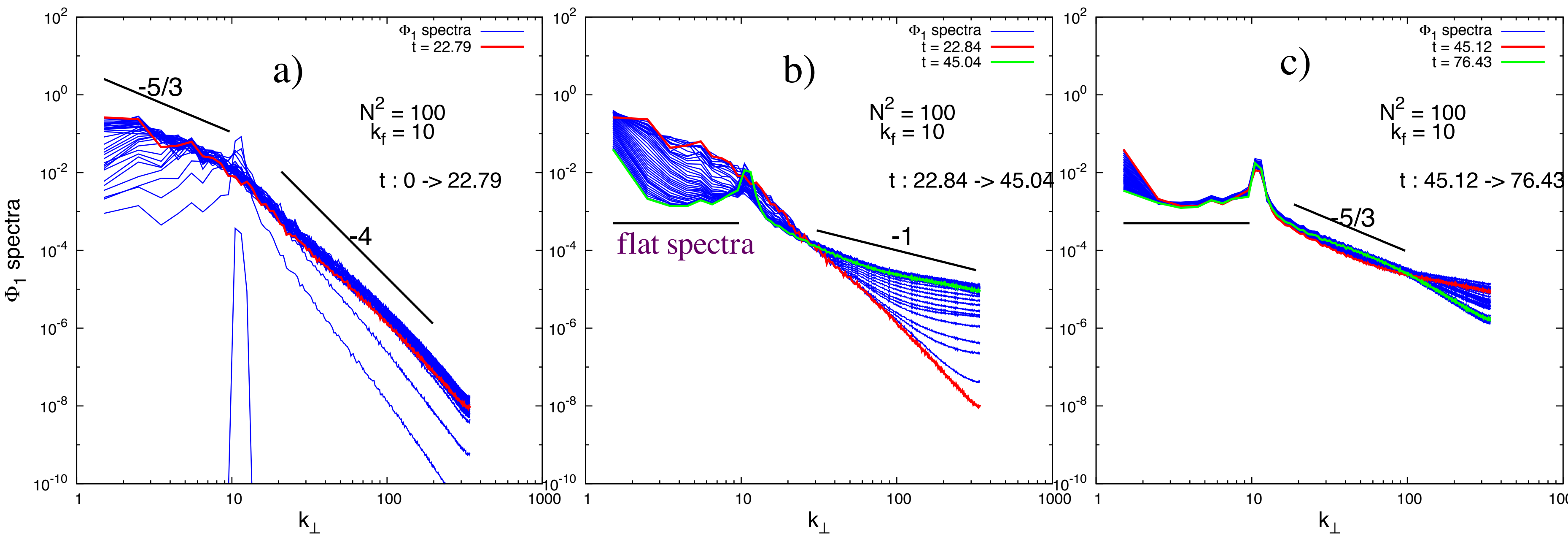




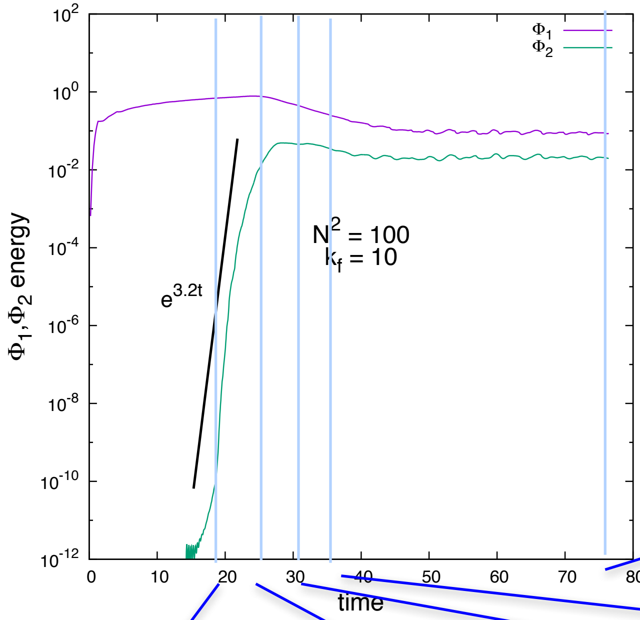
Growth of horizontal energy spectra

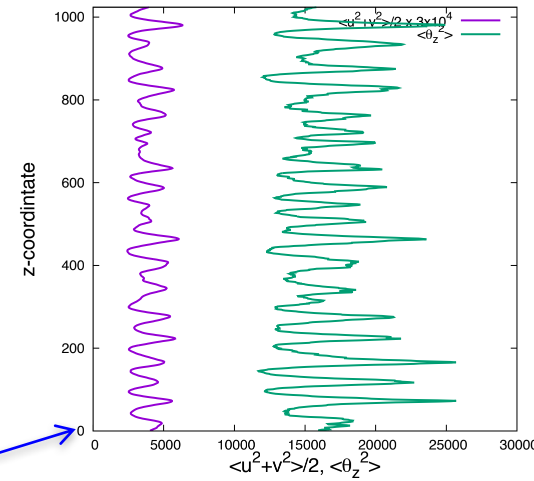


- Before stratification is switched, the flow field stays 2D-like and inverse cascade of energy is observed.
- After stratification is switched and gravity waves are developed, the low wave number energy is reduced.
- In the steady state, the high wave number part shows -5/3 spectrum.
- flat spectra in the low wave numbers were reported originally by Herring & Métais(1989) and recently verified by Marino, Mininni, Rosenberg & Pouquet (2014).



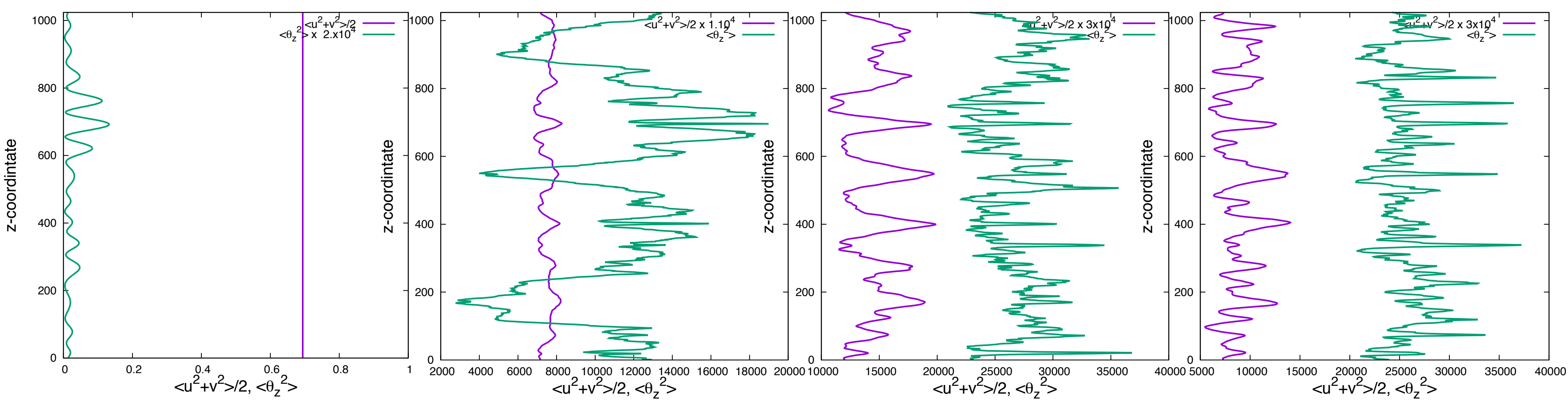
Development of temperature fluctuations





horizontal energy

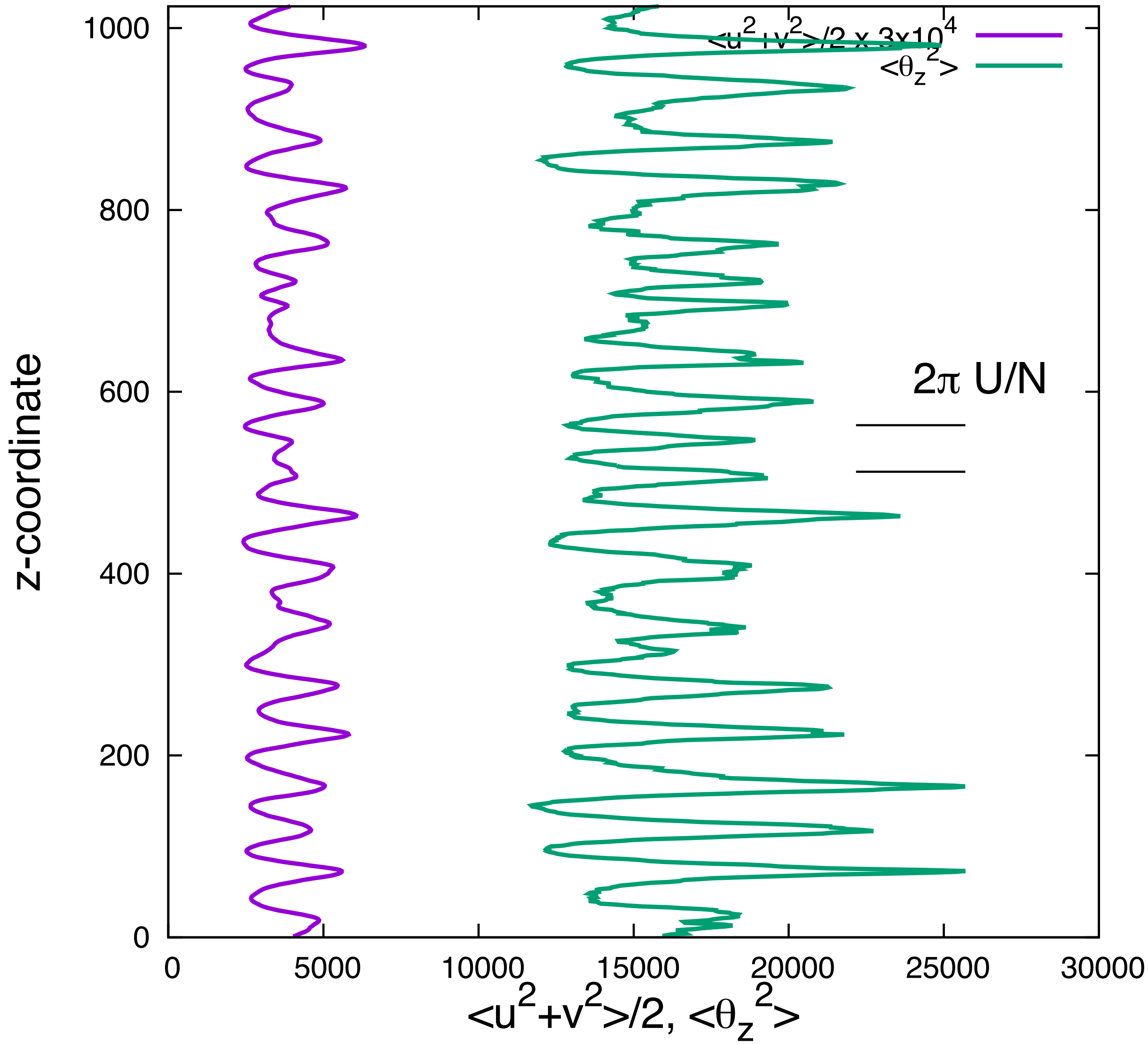
$\partial\theta/\partial z$



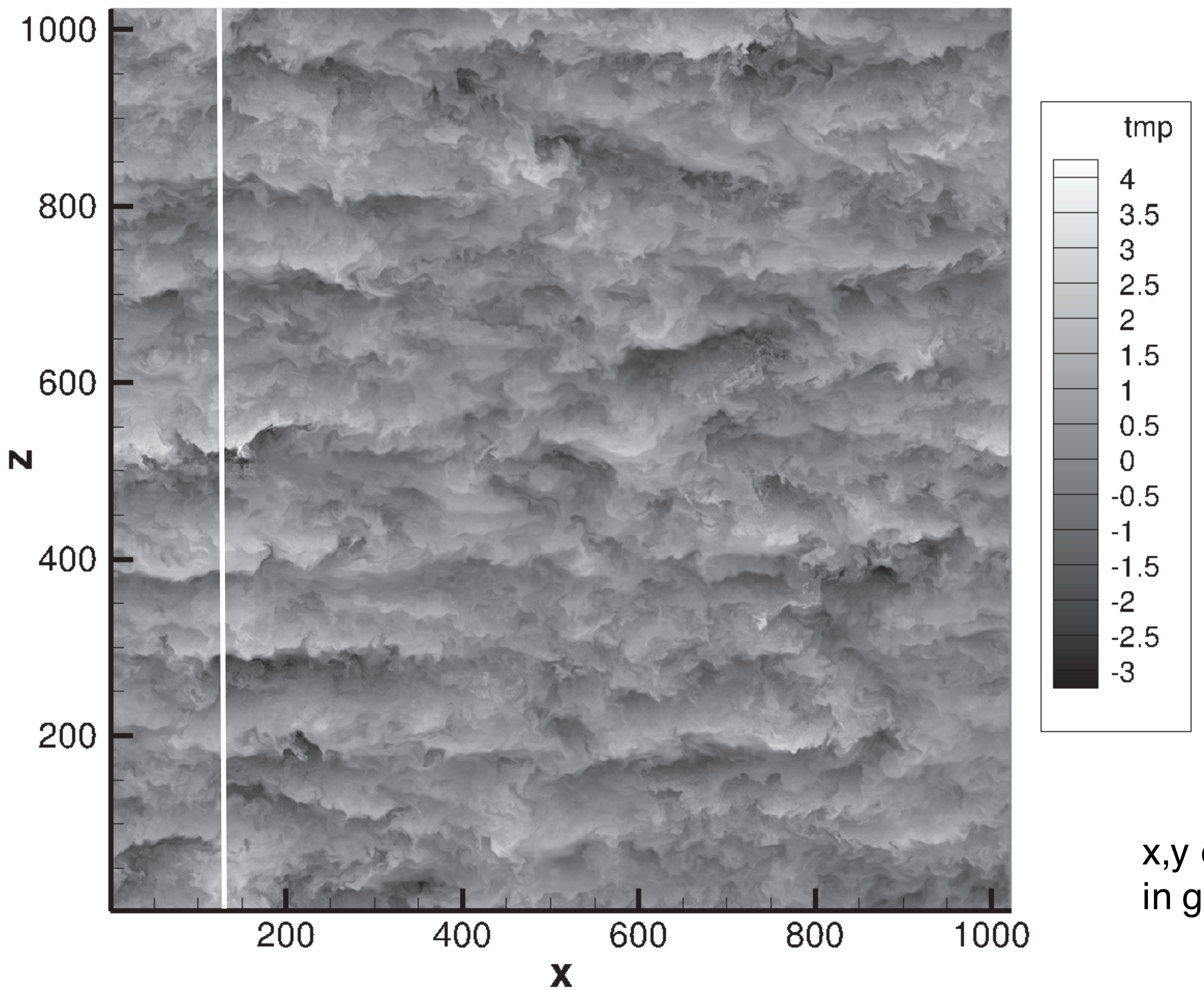
average of horizontal energy and vertical gradient of θ in horizontal planes

Horizontal kinetic energy and vertical derivative of θ

($k_f = 10$, $N^2 = 100$, $t = 76.5087$)

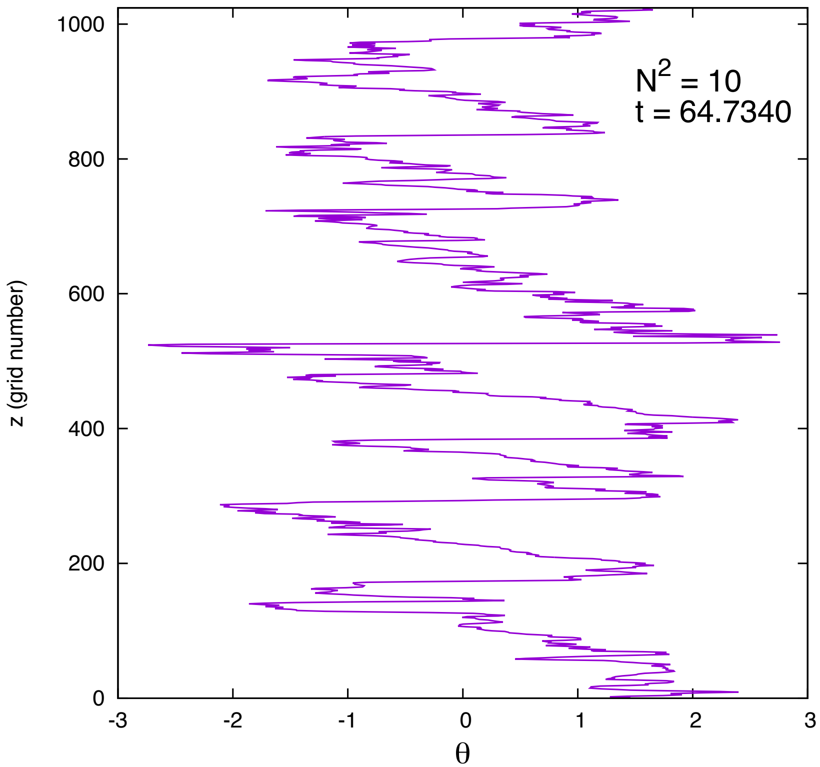


Temperature fluctuations in a vertical plane
(N^2=10, 1024^3 grid points)

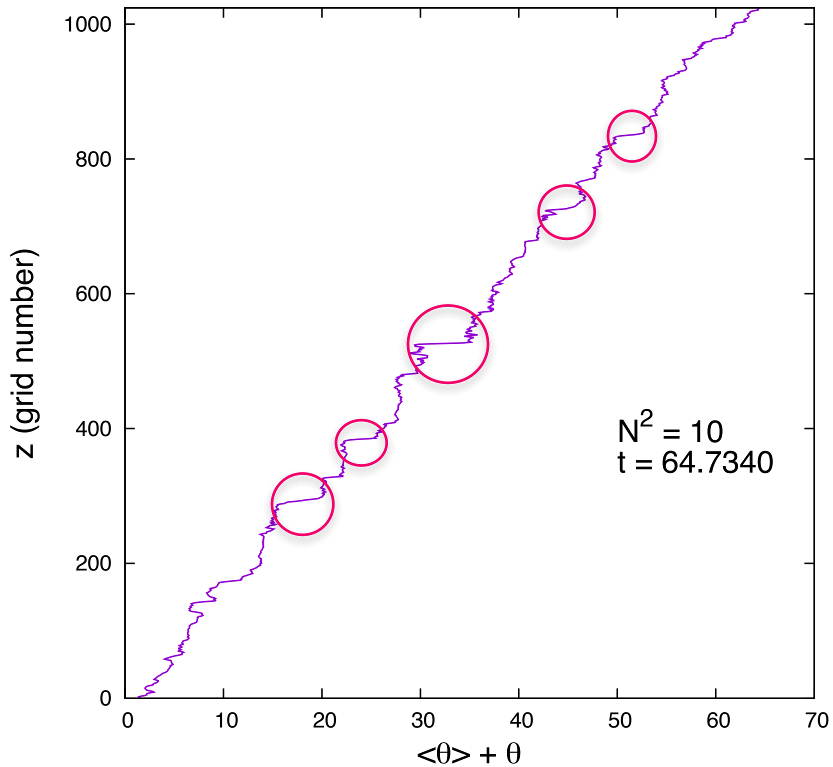


x,y coordinates
in grid numbers

Fluct. & total temperature along the white line at x=132

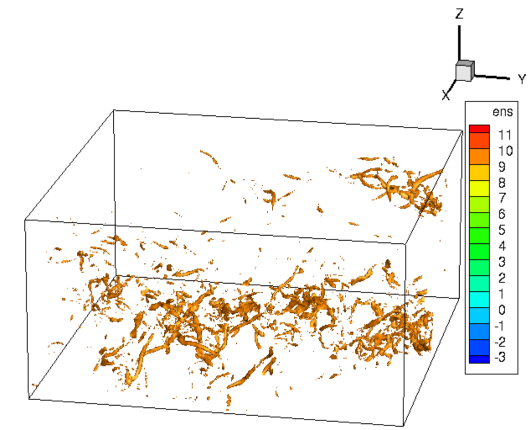


sew-tooth wavy jumps

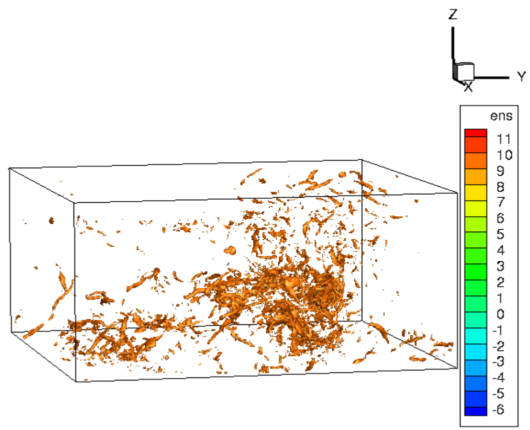


cliff-ramp structures

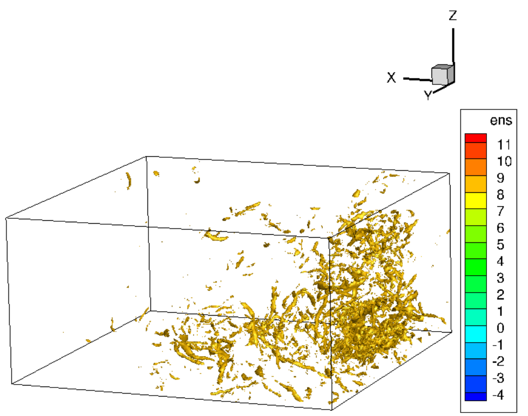
Search for the structures relating to the jumps



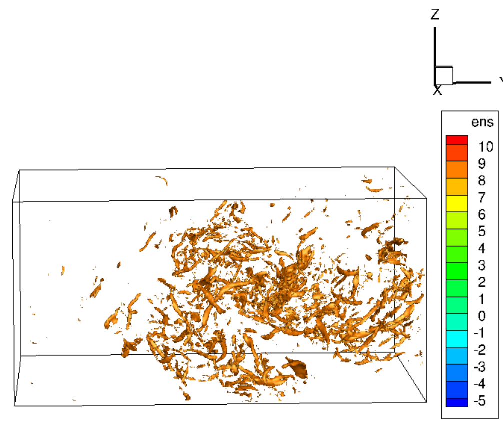
(x,y,z) = (101, 413, 251)



(x,y,z) = (187, 514, 300)

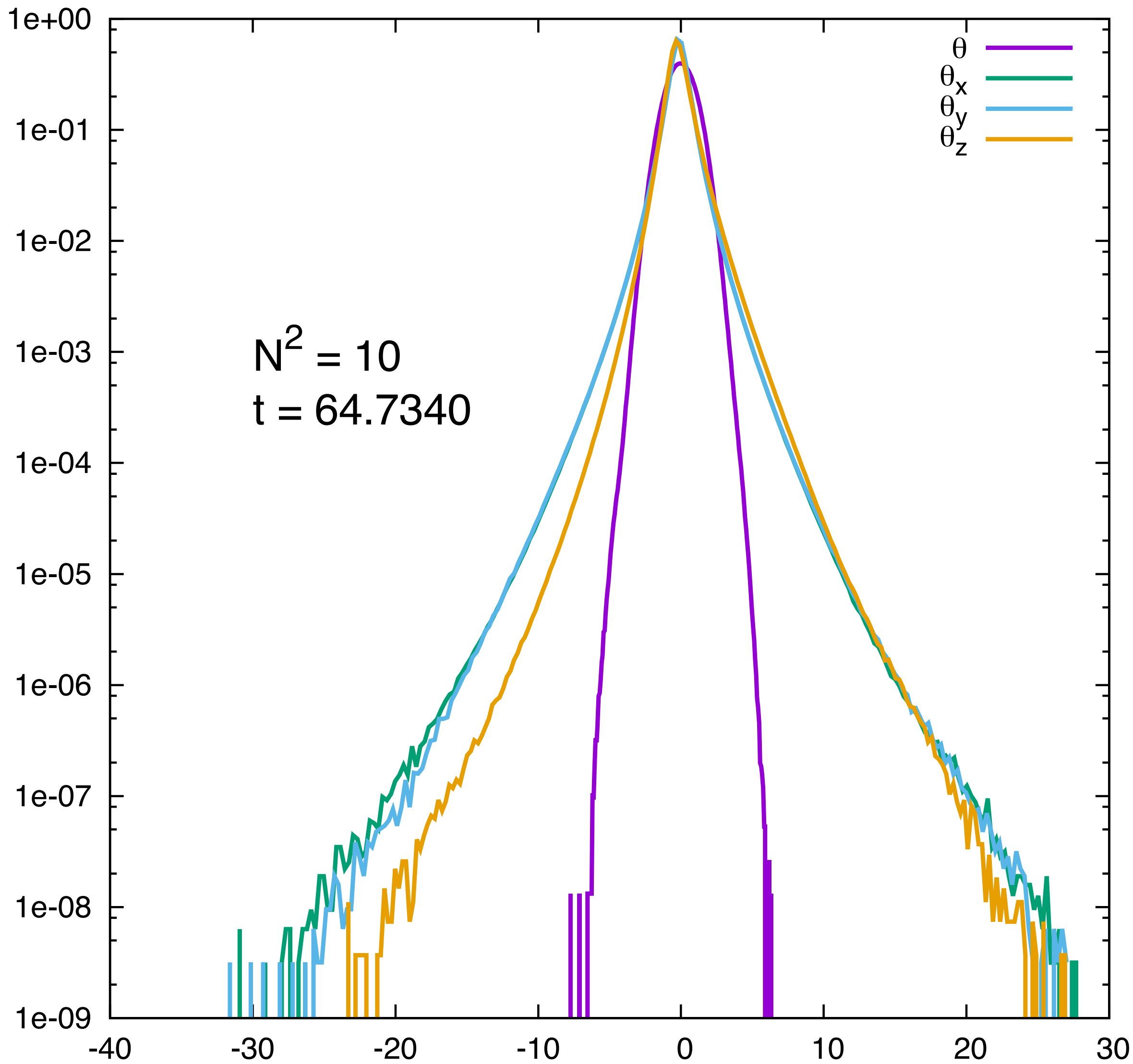


(x,y,z) = (300, 566, 300)



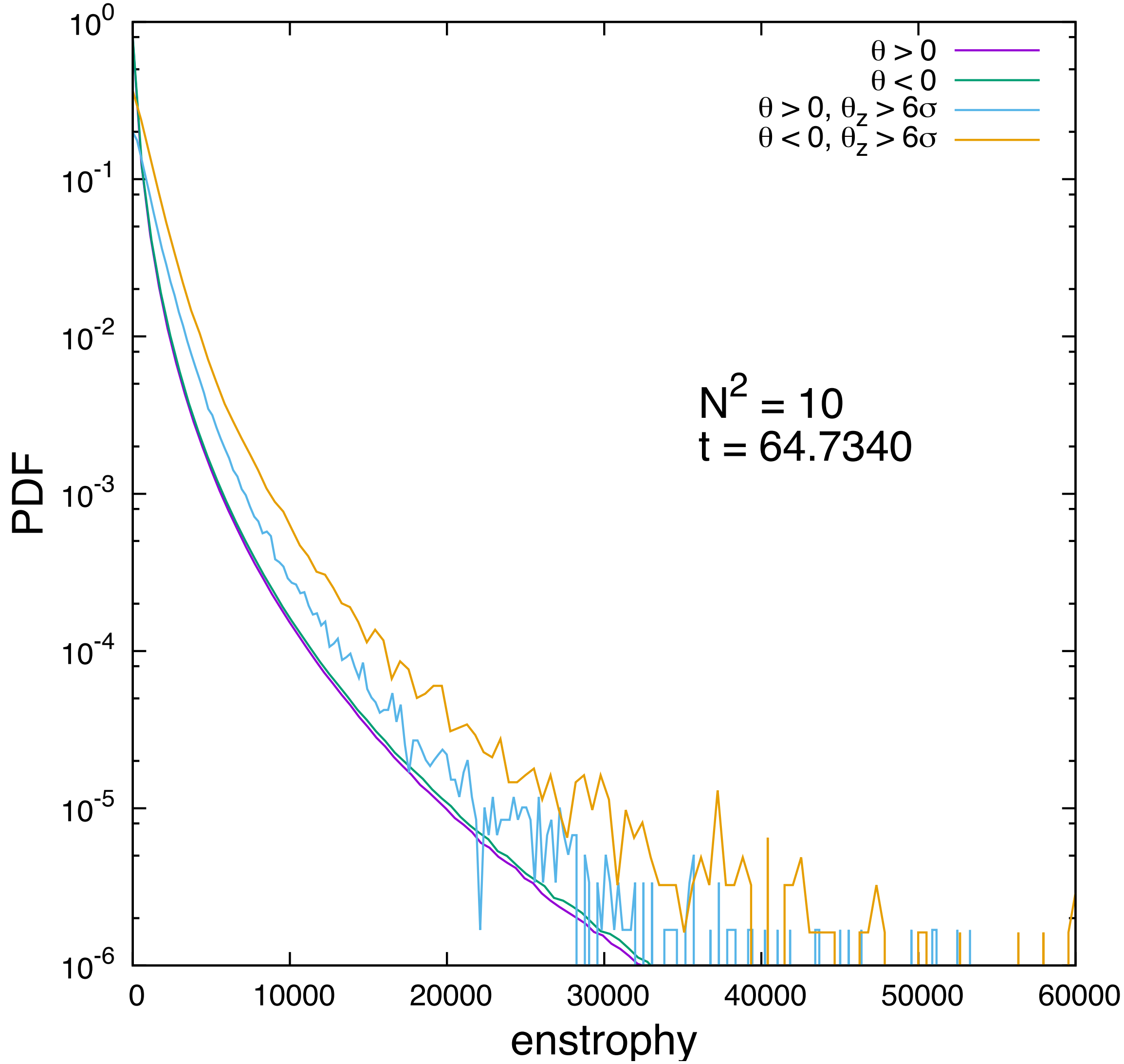
(x,y,z) = (650, 256, 500)

PDFs of temp. fluctuations and its derivatives



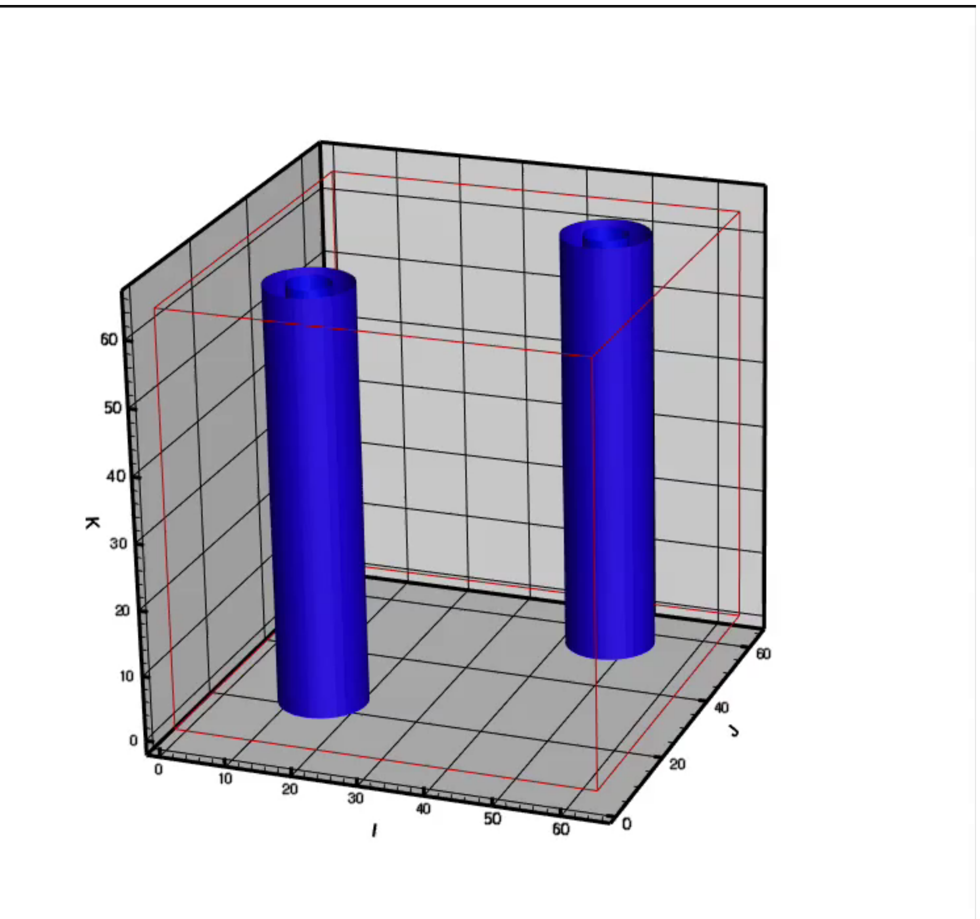
- PDF of θ_z is skewed.

Conditional PDF of enstrophy



- In the region of strong vertical gradient of temp. fluctuations, enstrophy is stronger in the cooler temperature than the warmer.

Model for tilted pancakes



Low-Froude number limit model

Riley, Metcalfe, Weissman (1981)

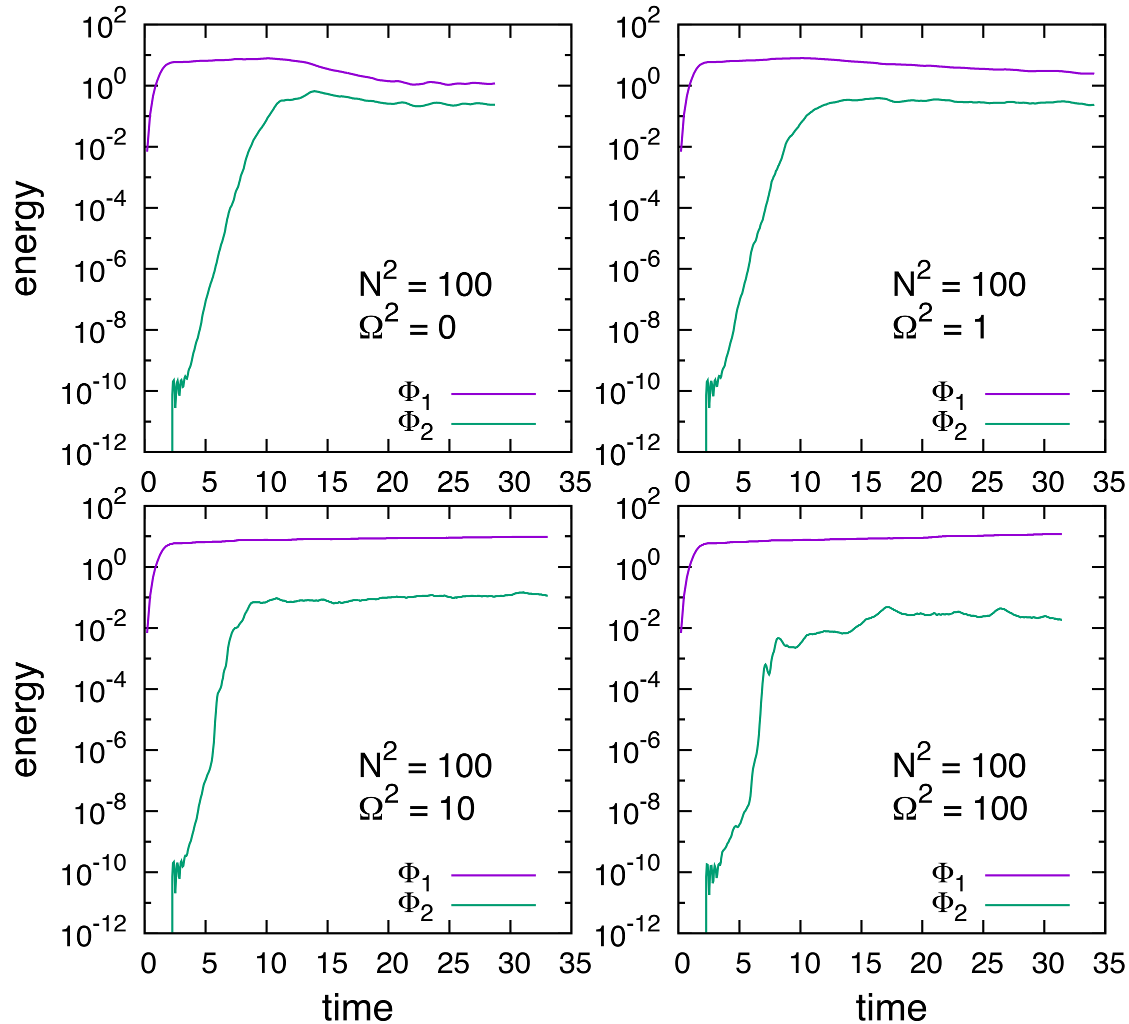
$$\begin{cases} \dfrac{\partial \vec{u}_H}{\partial t} + \vec{u}_H \cdot \nabla_H \vec{u}_H \\ \qquad = -\nabla_H p + \dfrac{1}{\mathrm{Re}} \left(\Delta_H + \dfrac{\partial^2}{\partial z^2} \right) \vec{u}_H \\ \mathrm{div}_{\mathrm{H}} \ \vec{u}_H = 0 \ , \quad w \equiv 0 \end{cases}$$

linear advection-diffusion eq.

(using a special initial condition)

Majda, A.J. & Grote M.J.
Model dynamics and vertical collapse in decaying strongly stratified flows.
Phys. Fluids **9** (1997) 2932-2940.

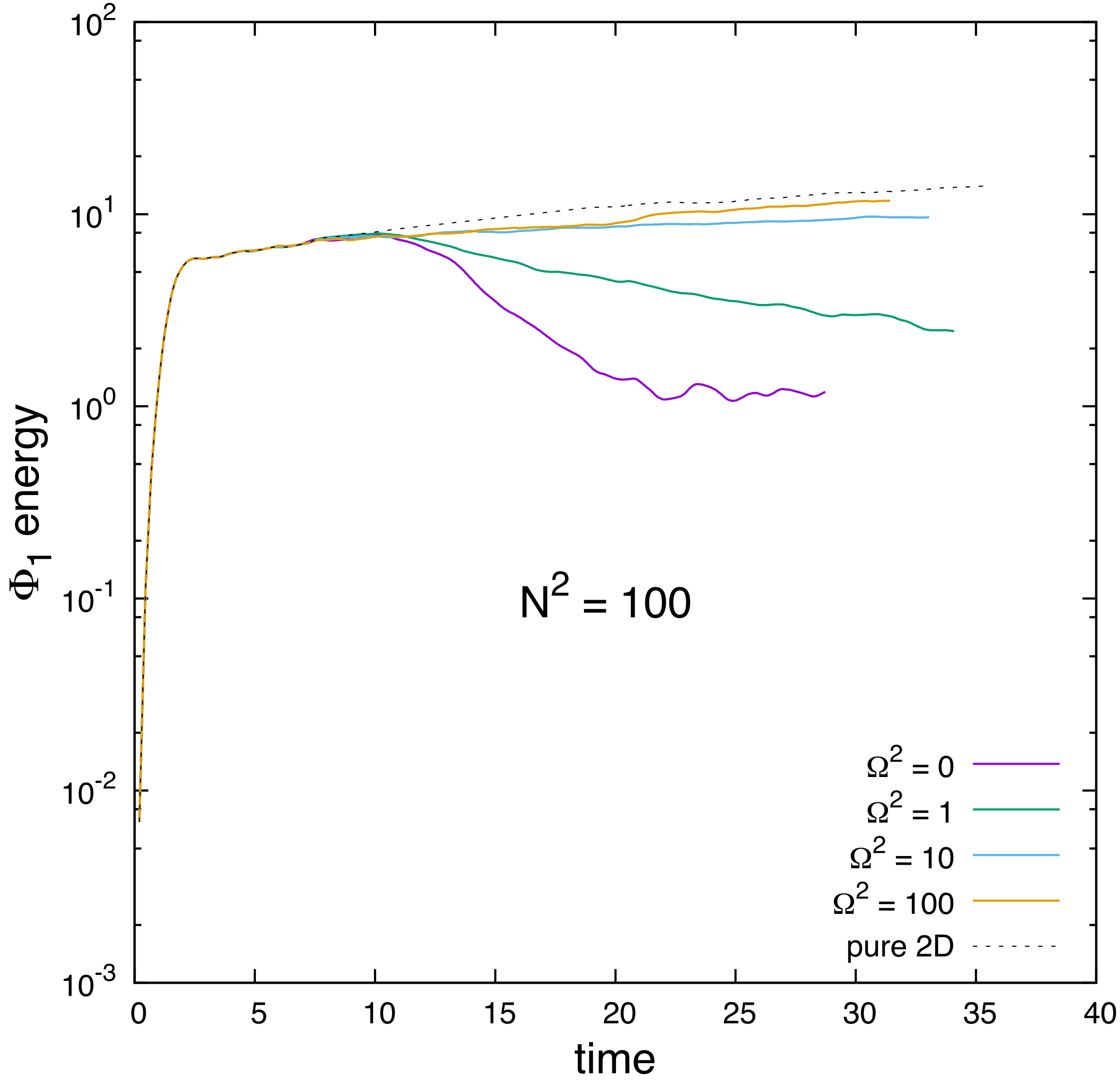
Effect of rotation (preliminary results, 512^3)
($k_f = 4$, pure 2D forcing)



Quasi-Geostrophic state?

- With rotation, wave amplitude decreases but the growth rate remains similar.

Development of Φ_1 energy



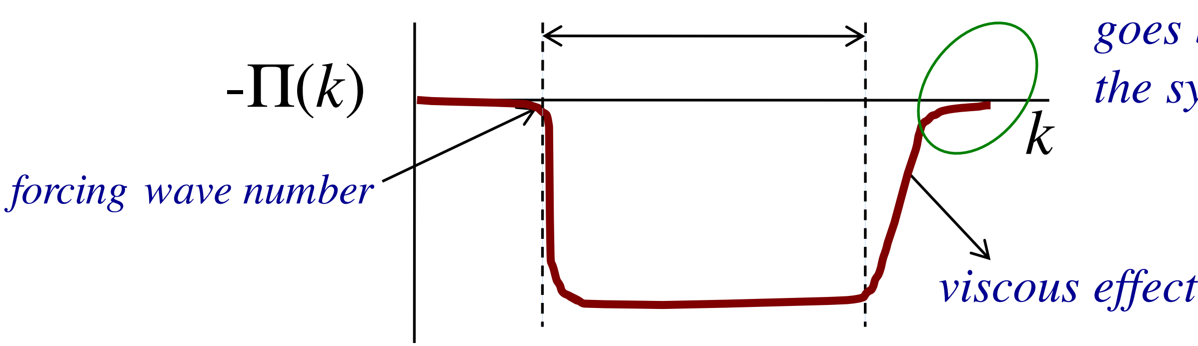
Spectral energy transfer by Kolmogorov

Kolmogorov (homogeneous isotropic) turbulence

$$\Pi(k) = -\int_0^k \hat{T}(k)\, dk \qquad \text{(flux function)}$$

spherical average of energy transfer function

basic idea: *constant flux* ≈ *power-law in spectrum* ≈ *inertial range*



goes back to 0 because of the symmetry of the wavespace

See for example:
T. Aoyama, T. Ishihara, Y. Kaneda, M. Yokokawa, K. Itakura & A. Uno *J. Phys. Soc. Jpn* **74**(2005) 3202-3212

Energy budget equation

From the equations of the motion in the Fourier expression,

$$(\partial_t + 2\nu k^2)\, E(\mathbf{k}) = \underline{T(\mathbf{k})} + \underline{B(\mathbf{k})} + \underline{F(\mathbf{k})}$$

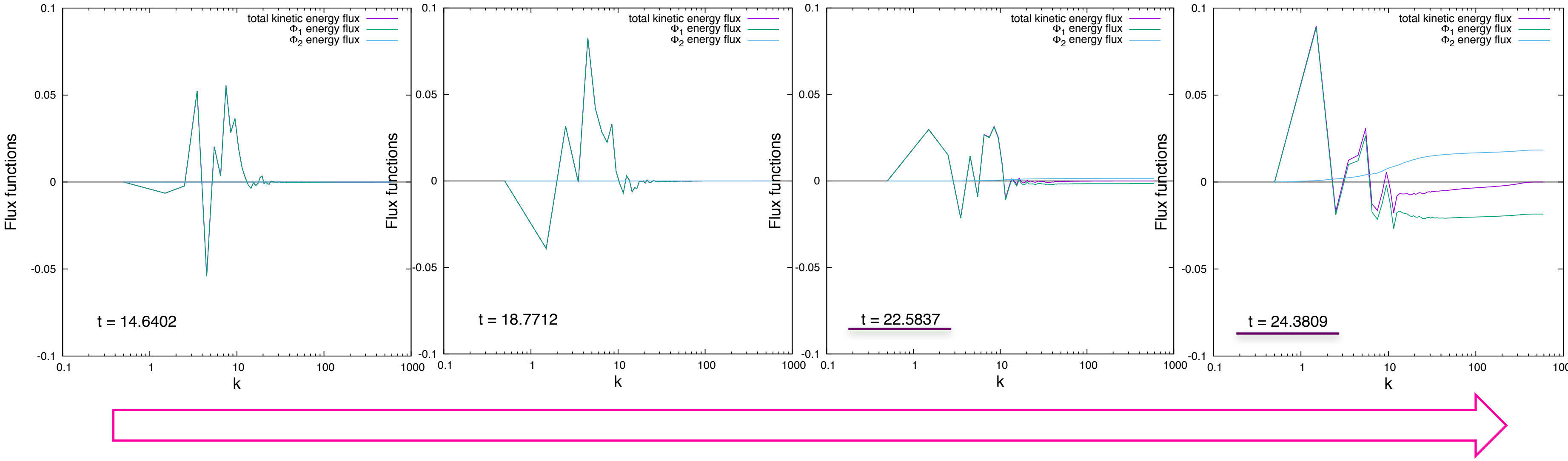
Nonlinear terms Buoyancy terms Forcing terms

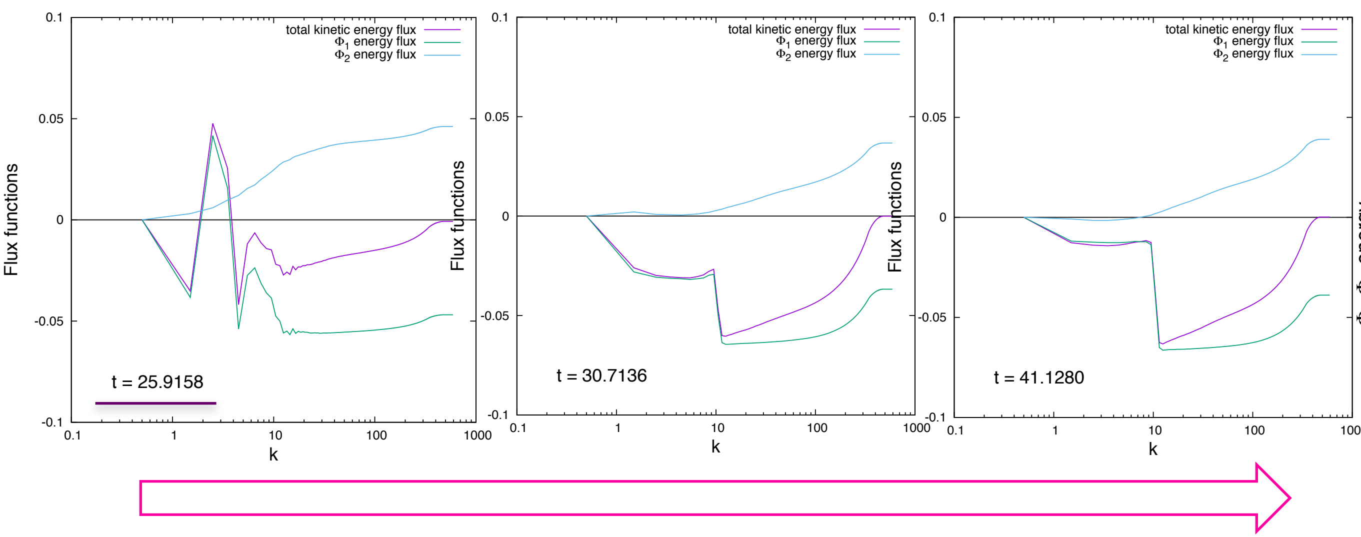
$$k^2 = |\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$$

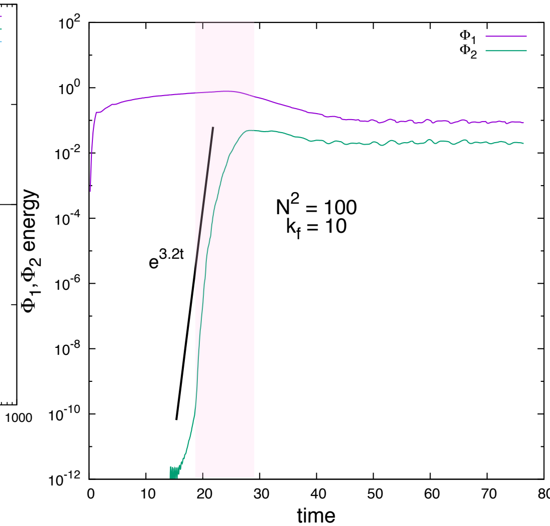
$$\begin{cases} T(\mathbf{k}) = 2P_{ij}(\mathbf{k})\ \mathrm{Im}\left[k_m \tilde{u}_i^*(\mathbf{k}) \left(\tilde{u}_j \otimes \tilde{u}_m\right)(\mathbf{k})\right] \\ B(\mathbf{k}) = 2P_{i3}(\mathbf{k})\ \mathrm{Re}\left[\tilde{\theta}(\mathbf{k}) \tilde{u}_i^*(\mathbf{k})\right] \\ F(\mathbf{k}) = 2P_{ij}(\mathbf{k})\ \mathrm{Re}\left[\tilde{f}_j(\mathbf{k}) \tilde{u}_i^*(\mathbf{k})\right] \end{cases}$$

($P_{ij} = \delta_{ij} - k_i k_j / |\mathbf{k}|^2$: Projection operator)

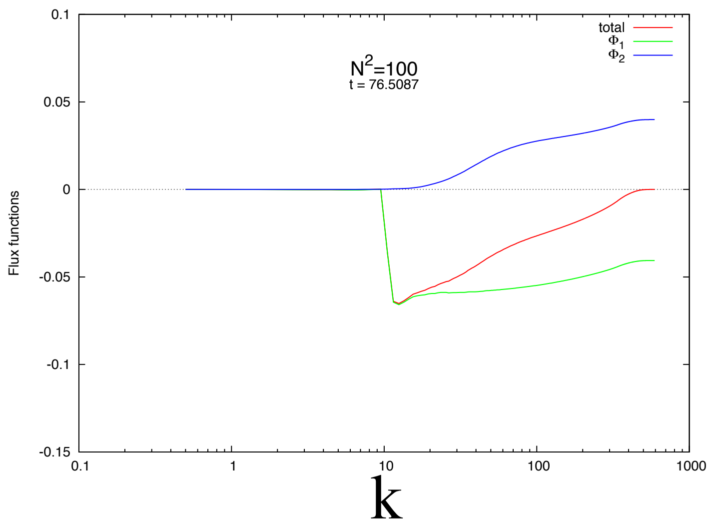
transition of the flux function

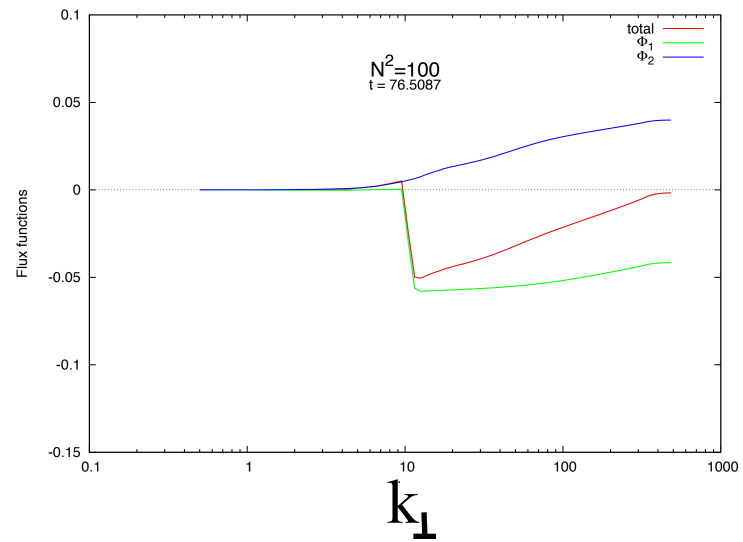


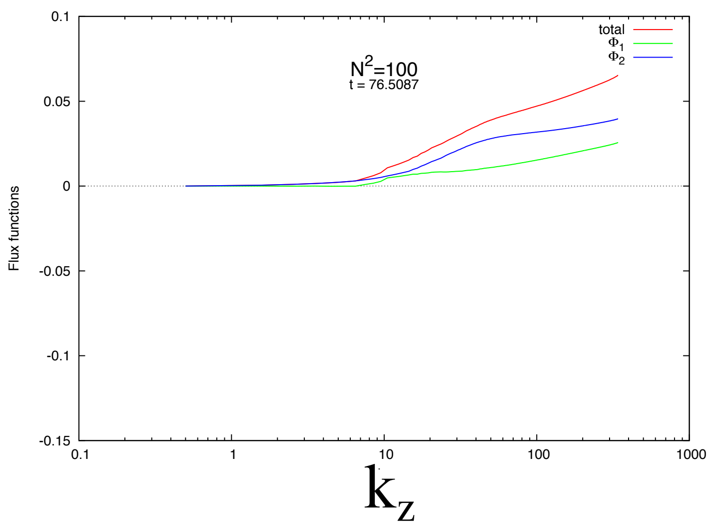




Energy flux functions with different wavenumbers (interaction between Φ_1 and Φ_2)

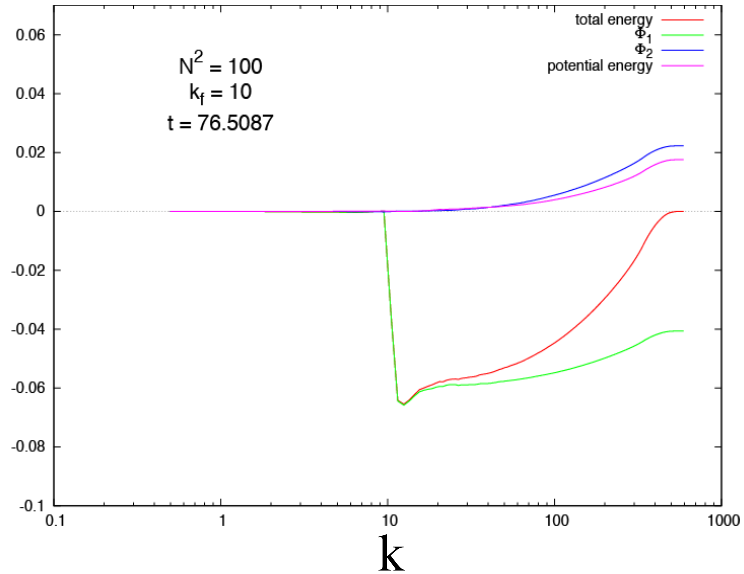


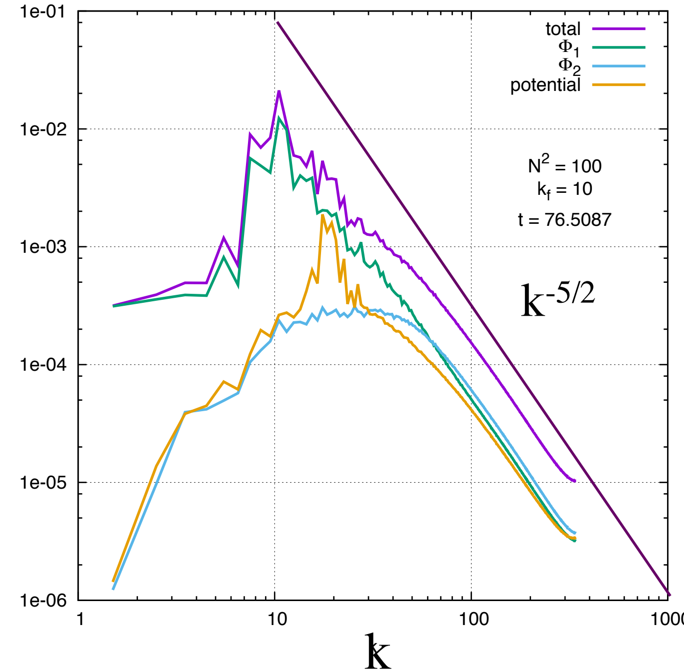




- Flux functions in terms $k_\perp$ and k_z do not go back to 0 in high wave numbers. (Because energy conservation is not hold.)
- Gravity waves (Φ_2) receive energy from Φ_1 in the high wave number region and contribute for energy dissipation.

Energy flux and spectrum



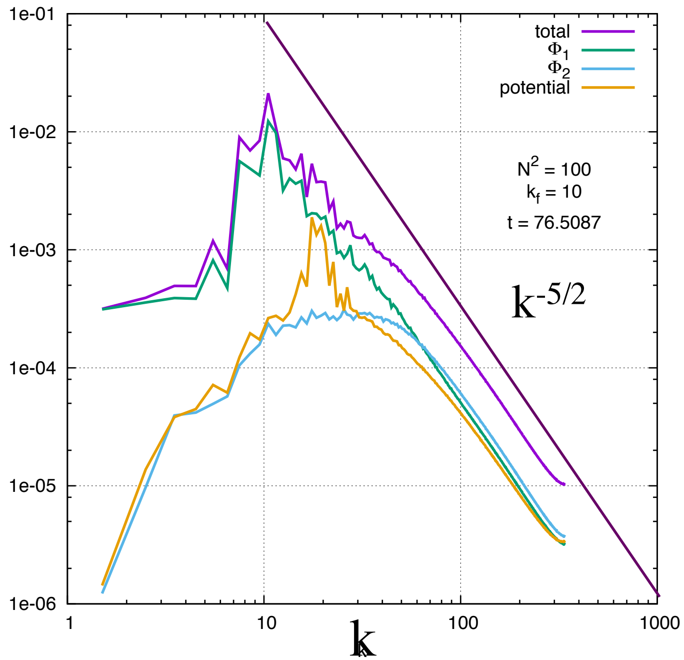


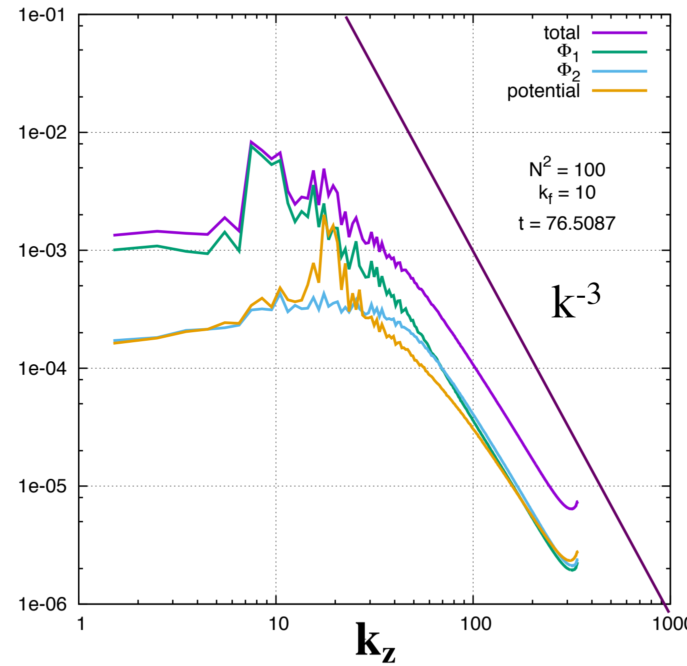
- Energy flux for Φ_1 takes nearly a constant value in the high wave number region.
- Φ_1 energy, Φ_2 energy, potential energy altogether satisfy conservation of energy.
- Corresponding energy spectrum satisfies a certain constant power-law. (Kolmogorov theory).

Problems:

- $\boldsymbol{\Phi_1}$, $\boldsymbol{\Phi_2}$ both tend to be 0 in the region of the high wave numbers
- There is no theoretical reasons for $k^{-5/2}$.

Effects of anisotropy





- k-spectra look like k_z-spectra in particular at high wave numbers. (perhaps because energy is distributed in the polar regions ($k_z >> 1$) due to the layer structures.
- $E(k_z) \sim CN^2 k_z^{-3}$ is known as the saturation spectrum and often observed in ocean science.

kinetic energy spectrum

Summary

- Horizontal layers develop quickly as the wave component grows.
- Rotation decreases the amplitude of waves keeping the growth rate similar.
- Kolmogorov's cascade picture needs to be modified because of the anisotropy.